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ORIGINAL ARTICLES

An Efficient Legendre Wavelet-Based Approximation Method for a Few Newell–Whitehead and Allen–Cahn Equations

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Abstract In this paper, a wavelet-based approximation method is introduced for solving the Newell–Whitehead (NW) and Allen–Cahn (AC) equations. To the best of our knowledge, until now there is no rigorous Legendre wavelets solution has been reported for the NW and AC equations. The highest derivative in the differential equation is expanded into Legendre series, this approximation is integrated while the boundary conditions are applied using integration constants. With the help of Legendre wavelets operational matrices, the aforesaid equations are converted into an algebraic system. Block pulse functions are used to investigate the Legendre wavelets coefficient vectors of nonlinear terms. The convergence of the proposed methods is proved. Finally, we have given some numerical examples to demonstrate the validity and applicability of the method.

Keywords Newell–Whitehead equation · Allen–Cahn equation · Operational matrices · Legendre wavelets · Laplace transform method · Homotopy analysis method

Introduction

Wavelet analysis, as a relatively new and emerging area in Applied Mathematical Research, has received considerable attention in dealing with PDEs (Hariharan et al. 2009; Hariharan and Kannan 2009a; Hariharan and Kannan 2010a; Hariharan and Kannan 2010b; Hariharan 2013; Jafari et al. 2011; Yang 2013; Heydari et al. 2012; Yin et al. 2012). In recent years, nonlinear reaction diffusion equations (NLRDE) have been used as a basis for a wide variety of models, for the special spread of gene in population (Hariharan and Kannan 2009a) and for chemical wave propagation (Babolian and Saeidian 2009).

Consider the general nonlinear parabolic equation is of the form

$$U_t = U_{xx} + aU + bU^n \tag{1}$$

where a and b are real constants. For $a = 1$, $b = -1$ and $n = 3$, Eq. (1) becomes Allen–Cahn (AC) equation. It arises in many scientific applications such as mathematical biology, quantum mechanics and plasma physics (Voigt 2001; Hariharan and Kannan 2009b). If for $n = 3$ and $b = -b$, then Eq. (1) becomes the Newell–Whitehead (NW) equation. This equation describes the dynamical behavior near the bifurcation point for the Rayleigh–Benard convection of binary fluid mixtures (Hariharan and Kannan 2010a; Nourazar et al. 2011; Schneider 1994; Babolian and Saeidian 2009; Kheiri et al. 2011; Ezzati and Shakibi 2011; Hammouch and Mekkaoui 2013).

In recent years, wavelet transforms have found their way into many different fields in science and engineering. Moreover, wavelet transform methods establish a connection with fast numerical algorithms.

Wavelet theory possesses many useful properties, such as Compact support, orthogonality, dyadic, orthonormality and multi-resolution analysis (MRA). In recent years, the AC equations play a significant role in fluid dynamics and many scientific applications. Recently, many new approaches to NLRDEs have been proposed, for example, the Adomian decomposition method (Ezzati and Shakibi 2011), homotopy perturbation method (Nourazar et al. 2011), homotopy analysis method (Hariharan 2013; Kheiri

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et al. 2011). A more comprehensive list of references about the NW and Allen–Cahn equation (ACE) and its applications in engineering can be found in (Hariharan and Kannan 2009a; Hariharan and Kannan 2010a; Schneider 1994; Voigt 2001; Kheiri et al. 2011; Hariharan and Kannan 2009b; Ezzati and Shakibi 2011). Recently, Hariharan and Rajaraman (2013) established a new coupled wavelet-based method applied to the NLRDE arising in mathematical chemistry. Yin et al. (2013) introduced a wavelet based hybrid method for solving Klein–Gordan equations.

In the numerical analysis, wavelet based methods and hybrid methods become important tools because of the properties of localization. In wavelet based methods, there are two important ways of improving the approximation of the solutions: increasing the order of the wavelet family and the increasing the resolution level of the wavelet. There is a growing interest in using various wavelets (Razzaghi and Yousefi 2001; Parsian 2005; Razzaghi and Yousefi 2000; Yousefi 2006; Mohammadi and Hosseini 2011; Maleknejad and Sohrabi 2007; Hariharan et al. 2009; Hariharan and Kannan 2009a; Hariharan and Kannan 2010a; Hariharan and Kannan 2010b; Jafari et al. 2011; Yang 2013; Heydari et al. 2012; Yin et al. 2012; Khellat and Yousefi 2006) to study problems, of greater computational complexity. Among the wavelet transform families the Haar and Legendre wavelets deserve much attention. The basic idea of Legendre wavelet method is to convert the PDEs to a system of algebraic equations by the operational matrices of integral or derivative (Hariharan et al. 2009; Hariharan and Kannan 2009a; Hariharan and Kannan 2010a; Hariharan and Kannan 2010b). The main goal is to show how wavelets and MRA can be applied for improving the method in terms of easy implementability and achieving the rapidity of its convergence. Razzaghi and Yousefi (2001); Razzaghi and Yousefi 2000) introduced the Legendre wavelet method for solving variational problems and constrained optimal control problems. Hariharan et al. (Hariharan et al. 2009; Hariharan and Kannan 2009a; Hariharan and Kannan 2010a; Hariharan and Kannan 2010b) had introduced the diffusion equation, convection–diffusion equation, Reaction–diffusion equation, non linear parabolic equations and Fisher's equation by the Haar wavelet method. Mohammadi and Hosseini (Mohammadi and Hosseini 2011) had showed a new Legendre wavelet operational matrix of derivative in solving singular ordinary differential equations. Jafari et al. (2011) had solved the fractional differential equations by the Lagendre wavelet method. Parsian (2005) introduced two dimension Legendre wavelets and operational matrices of integration. In recent years, many analytical/approximation methods have been proposed for solving the NW and AC equations.

In this work, we have applied a wavelet-based coupled method (LLWM) which combines the Laplace transform method and the Legendre wavelets method for the numerical solutions of the NW and AC equations.

This paper is organized as follows: basic definitions of wavelets, Legendre wavelets and their properties are described in Legendre Wavelets and Properties Section. Then, the method of solution of the NW and AC equations by the LLWM is presented in Method of Solution Section. In Convergence Analysis and Error Estimation Section, the convergence analysis is described. In Illustrative Examples Section, several numerical examples are presented to demonstrate the effectiveness of the proposed method. Concluding remarks are given in Conclusion Section.

Legendre Wavelets and Properties

Wavelets

Wavelets are the family of functions which are derived from the family of scaling function $\{\emptyset_{j,k:}\ k \in Z\}$ where:

$$\emptyset(x) = \sum\nolimits_k a_k\, \emptyset(2x - k) \quad (2)$$

For the continuous wavelets, the following equation can be represented:

$$\Psi_{a,b}(x) = |a|^{\frac{-1}{2}} \Psi\left(\frac{x-b}{a}\right) \qquad a, b \in R,\ a \neq 0. \quad (3)$$

where a and b are dilation and translation parameters, respectively, such that $\Psi(x)$ is a single wavelet function.

The discrete values are put for a and b in the initial form of the continuous wavelets, i.e.:

$$a = a_0^{-j},\ a_0 > 1,\ b_0 > 1, \quad (4)$$

$$b = kb_0 a_0^{-j},\ j, k \in Z. \quad (5)$$

Then, a family of discrete wavelets can be constructed as follows:

$$\Psi_{j,k} = |a_0|^{\frac{1}{2}} \Psi(2^j x - k), \quad (6)$$

So, $\Psi_{j,k}(x)$ constitutes an orthonormal basis in L^2 (R), where $\Psi(x)$ is a single function.

Legendre Wavelets

The Legendre wavelets are defined by

$$\Psi_{nm}(t)=\begin{cases}\sqrt{m+\frac{1}{2}}2^{\frac{k}{2}}L_m(2^k t-\widehat{n}), & for \quad \frac{\widehat{n}-1}{2^k}\le t\le\frac{\widehat{n}+1}{2^k},\\ 0, & otherwise\end{cases} \tag{7}$$

where $m = 0, 1, 2, \ldots, M - 1$, and $n = 1, 2, \ldots, 2^{k-1}$. The coefficient $\sqrt{m+\frac{1}{2}}$ is for orthonormality, then, the wavelets $\Psi_{k,m}(x)$ form an orthonormal basis for $L^2[0, 1]$ [] In the above formulation of Legendre wavelets, the Legendre polynomials are in the following way:

$$p_0 = 1,$$

$$p_1 = x,$$

$$p_{m+1}(x)=\frac{2m+1}{m+1}x\,p_m(x)-\frac{m}{m+1}p_{m-1}(x). \tag{8}$$

and $\{p_{m+1}(x)\}$ are the orthogonal functions of order m, which is named the well-known shifted Legendre polynomials on the interval [0, 1] Note that, in the general form of Legendre wavelets, the dilation parameter is $a = 2^{-k}$ and the translation parameter is $b = n\, 2^k$.

Block Pulse Functions (BPFs) (Yin et al. 2013)

The block pulse functions form a complete set of orthogonal functions which defined on the interval [0, b] by

$$b_i(t)=\begin{cases}1, & \frac{i-1}{m}b\le t<\frac{i}{m}b\\ 0 & elsewhere\end{cases} \tag{9}$$

for $i = 1, 2, \ldots, m$. It is also known that for any absolutely integrable function $f(t)$ on [0, b] can be expanded in block pulse functions:

$$f(t)\cong \boldsymbol{\xi^T B_m(t)} \tag{10}$$

$$\boldsymbol{\xi^T}=[f_1,f_2,\ldots,f_m], B_m(t)=[b_1(t),b_2(t),\ldots b_m(t)] \tag{11}$$

where f_i are the coefficients of the block-pulse function, given by

$$f_i=\frac{m}{b}\int_0^b f(t)b_i(t)dt \tag{12}$$

Remark 1: Let A and B are two matrices of $m \times m$, then $A\otimes B=(a_{ij}\times b_{ij})_{mm}$.

Lemma 1: *Assuming f(t) and g(t) are two absolutely integrable functions, which can be expanded in block pulse function as f(t) = FB(t) and g(t) = GB(t) respectively, then we have*

$$f(t)g(t)-FB(t)B^T(t)G^T=HB(t) \tag{13}$$

where $H=F\otimes G$.

Approximating the Nonlinear Term (Yin et al. 2013)

The Legendre wavelets can be expanded into m-set of block-pulse functions as

$$\Psi(t)=\emptyset_{m\times m}B_m(t) \tag{14}$$

Taking the collocation points as following

$$t_i=\frac{i-\mathbf{1}/2}{2^{k-\mathbf{1}}M},\quad i=1,2,\ldots,2^{K-1}M \tag{15}$$

The m-square Legendre matrix $\emptyset_{m\times m}$ is defined as

$$\emptyset_{m\times m}\cong[\Psi(t_1)\Psi(t_2)\ldots\Psi(t_{2^{k-1}M})] \tag{16}$$

The operational matrix of product of Legendre wavelets can be obtained by using the properties of BPFs, let $f(x,t)$ *and* $g(x,t)$ are two absolutely integrable functions, which can be expanded by Legendre wavelets as $f(x,t)=\Psi^T(x)F\Psi(t)$ and $g(x,t)=\Psi^T(x)G\Psi(t)$ respectively.

Then

$$f(x,t)=\Psi^T(x)F\Psi(t)=B^T(x)\emptyset^T_{mm}F\emptyset_{mm}B(t), \tag{17}$$

$$g(x,t)=\Psi^T(x)G\Psi(t)=B^T(x)\emptyset^T_{mm}G\emptyset_{mm}B(t), \tag{18}$$

and

$$F_b=\emptyset^T_{mm}F\emptyset_{mm}, G_b=\emptyset^T_{mm}G\emptyset_{mm}, H_b=F_b\otimes G_b.$$

$$f(x,t)g(x,t)=B^TH_bB(t),$$

$$=B^T(x)\emptyset^T_{mm}inv\left(\emptyset^T_{mm}\right)H_b inv\left(inv\left(\emptyset^T_{mm}\right)H_b inv(\emptyset_{mm})\right)\emptyset_{mm}B(t)$$

$$=\Psi^T(x)H\Psi(t) \tag{19}$$

where $H = inv(\emptyset_{mm}^{T})H_b inv((\emptyset_{mm}))$

Function Approximation

A given function $f(x)$ with the domain [0, 1] can be approximated by:

$$f(x) = \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} c_{k,m}\Psi_{k,m}(x) = C^T.\Psi(x). \tag{20}$$

Here C and Ψ are the matrices of size $\left(2^{j-1}\mathrm{M} \times 1\right)$.

$$C = \left[c_{1,0}, c_{1,1}, \ldots c_{1,M-1}, c_{2,0}, c_{2,1}, \ldots c_{2,M-1}, \ldots c_{2^{j-1},1}, \ldots c_{2^{j-1},M-1}\right]^T \tag{21}$$

$$\Psi(x) = [\Psi_{1,0}, \Psi_{1,1}, \Psi_{2,0}, \Psi_{2,1}, \ldots \Psi_{(2,M-1)}, \ldots \Psi_{(2^{(j-1)},M-1)}]^{\mathrm{T}}. \tag{22}$$

Method of Solution

Solving the Cahn–Allen Equation by the LLWM

We consider the AC equation

$$U_t = \varepsilon U_{xx} + U - U^3 \tag{23}$$

Taking Laplace transform on both sides of Eq. (23), we get

$$sL(U) - U(x,0) = L[\varepsilon U_{xx} + U - U^3] \tag{24}$$

$$sL(U) = U(x,0) + L[\varepsilon U_{xx} + U - U^3] \tag{25}$$

$$L(U) = \frac{U(x,0)}{s} + \frac{1}{s}L[\varepsilon U_{xx} + U - U^3] \tag{26}$$

Taking inverse Laplace transform to Eq. (26) we get

$$U(x,t) = U(x,0) + L^{-1}\left(\frac{1}{s}L[\varepsilon U_{xx} + U - U^3]\right) \tag{27}$$

Because

$$L^{-1}\left[\frac{1}{s}L(t^n)\right] = L^{-1}\left(\frac{n!}{s^{n+2}}\right) = \frac{1}{n+1}t^{n+1}; (n = 0, 1, 2, \ldots) \tag{28}$$

We have

$$L^{-1}[s^{-1}L()] = \int_0^t (.)dt \tag{29}$$

From Eq. (29), we obtain

$$U(x,t) = U(x,0) + L^{-1}\left(\frac{1}{s}L(\varepsilon U_{xx} + g(U))\right) \tag{30}$$

Where $g(U) = U - U^3$

$$U(x,t) = U(x,0) + L^{-1}\left(\frac{1}{s}L(\varepsilon U_{xx} + g(U))\right) \tag{31}$$

By using the Legendre wavelets method,

$$\left.\begin{aligned} U(x,t) &= C^T\psi(x,t) \\ U(x,0) &= S^T\psi(x,t) \\ g(U) &= G^T\psi(x,t) \end{aligned}\right\} \tag{32}$$

Substituting Eq. (32) in Eq. (27), we obtain

$$C^T = S^T + (\varepsilon C^T Dx^2 - G^T)P_t^2 \tag{33}$$

Here G^T has a nonlinear relation with C. When we solve a nonlinear algebraic system, we get the solution is more complex and large computation time. In order to overcome the above drawbacks, we introduce an approximation formula as follows:

$$U_{n+1} = U(x,0) + \Pi\left[\frac{\partial^2 U_n}{\partial x^2} + g(U_n)\right] \tag{34}$$

where $g(U) = \alpha U - \alpha U^2$

Expanding $u(x, t)$ by Legendre wavelets using the following relation

$$C_{n+1}{}^T = C_0{}^T + [C_n{}^T D_x{}^2 - G_n{}^T]P_t{}^2 \tag{35}$$

Solving the Newell–Whitehead equation by the LLWM

We consider the Nowell–Whitehead equation

$$U_t = U_{xx} + aU - bU^3 \tag{36}$$

Taking Laplace transform on both sides of Eq. (36), we get

$$sL(U) - U(x,0) = L[U_{xx} + aU - bU^3] \tag{37}$$

$$sL(U) = U(x,0) + L[U_{xx} + aU - bU^3] \tag{38}$$

$$L(U) = \frac{U(x,0)}{s} + \frac{1}{s}L[U_{xx} + aU - bU^3] \tag{39}$$

Taking inverse Laplace transform to Eq. (39) we get

$$U(x,t) = U(x,0) + L^{-1}\left(\frac{1}{s}L[U_{xx} + aU - bU^3]\right) \quad (40)$$

Because

$$L^{-1}\left[\frac{1}{s}L(t^n)\right] = L^{-1}\left(\frac{n!}{s^{n+2}}\right) = \frac{1}{n+1}t^{n+1}; (n = 0, 1, 2, \ldots) \quad (41)$$

We have

$$L^{-1}[s^{-1}L()] = \int_0^t (.)dt \quad (42)$$

From Eq. (42), we obtain

$$U(x,t) = U(x,0) + L^{-1}\left(\frac{1}{s}L(U_{xx} + g(U)))\right) \quad (43)$$

Where $g(U) = aU - bU^3$

$$U(x,t) = U(x,0) + L^{-1}\left(\frac{1}{s}L(U_{xx} + g(U)))\right) \quad (44)$$

By using the Legendre wavelets method,

$$\left.\begin{aligned} U(x,t) &= C^T\psi(x,t) \\ U(x,0) &= S^T\psi(x,t) \\ g(U) &= G^T\psi(x,t) \end{aligned}\right\} \quad (45)$$

Substituting Eq. (45) in Eq. (43), we obtain

$$C^T = S^T + (C^T Dx^2 - G^T)P_t^2 \quad (46)$$

Here G^T has a nonlinear relation with C. When we solve a nonlinear algebraic system, we get the solution is more complex and large computation time. In order to overcome the above drawbacks, we introduce an approximation formula as follows:

$$U_{n+1} = U(x,0) + \Pi\left[\frac{\partial^2 U_n}{\partial x^2} + g(U_n)\right] \quad (47)$$

where $g(U) = aU - bU^3$

Expanding $u(x, t)$ by Legendre wavelets using the following relation

$$C_{n+1}{}^T = C_0{}^T + [C_n{}^T D_x{}^2 - G_n{}^T]P_t{}^2$$

Convergence Analysis and Error Estimation (Yin et al. 2013)

$$U^* = U_0 + \Pi\left[U^*_{xx} + g(U^*)\right] \quad (48)$$

$$U_{n+1} = U_0 + \Pi\left[(U_n)_{xx} + g(U_n)\right] \quad (49)$$

Subtracting Eq. (48) from Eq. (49), we obtain

$$U_{n+1} - U^* = \Pi\left[(U_n - U^*)_{xx} + g(U_n) - g(U^*)\right] \quad (50)$$

Using Lipschitz condition, $\|g(U_n) - g(U^*)\| \le \gamma \|U_n - U^*\|$, we have

$$\|U_{n+1} - U^*\| \le \left\|\Pi(U_n - U^*)_{xx}\right\| + \|\Pi(g(U_n) - g(U^*))\| \quad (51)$$

$$\le \left\|\Pi(U_n - U^*)_{xx}\right\| + \gamma\|\Pi(U_n - U^*)\| \quad (52)$$

Let $U_{n+1} = C^T_{n+1}\psi(x,t)$

$$U^* = C^T\psi(x,t)$$

$$\in^T_{n+1} = C^T_{n+!} - C^T$$

Eq. (52) gives

$$\in^T_{n+1} \le \in^T_n \left\|D_x^2 P_t^2 + \gamma P_t^2\right\| \quad (53)$$

The following formula Eq. (54) can be obtained by using recursive relation.

$$\in_{n+1}{}^T \le \in_n{}^T \left\|D_x{}^2 P_t{}^2 + \gamma P_t{}^2\right\|^n \in_0 \quad (54)$$

When $\lim_{n\to\infty}\left\|D_x{}^2P_t{}^2 + \gamma P_t{}^2\right\|^n = 0$, the series solution of Eq. (23) using the LLWM converges to $u^*(x)$. By using the definitions of D_x and P_t, we can get the value of γ. Suppose $k = k' = 1$ and $M = M'$, the maximum element of D_x and P_t is $2\sqrt{(2M-1)(2M-3)}$ and 0.5 respectively.

Error Estimation

The accuracy of the proposed method LLWM is estimated by the following error function

$$E_J = \left|U_{exact} - U_{approx}\right|$$

Illustrative Examples

Example 5.1 We consider the AC equation of the form (Hariharan and Kannan 2010a; Voigt 2001; Hariharan and Kannan 2009b)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1 - u^2) \quad (55)$$

The exact solution in a closed form is given by

$$u(x,t) = \frac{1}{1 + e^{-\left(\frac{\sqrt{2}}{2}\right)\left[x + \left(\frac{3\sqrt{2}}{2}\right)t\right] + c_0}}, \quad (56)$$

where c_0 is integration constant.

The approximation formula for Eq. (55) as follows

$$U_{n+1} = U(x,0) + \Pi\left[\frac{\partial^2 U_n}{\partial x^2} + g(U_n)\right] \quad (57)$$

where $g(U) = aU - bU^3$

Expanding $u(x, t)$ by Legendre wavelets using the following relation

$$C_{n+1}{}^T = C_0{}^T + [C_n{}^T D_x{}^2 - G_n{}^T]P_t{}^2 \quad (58)$$

Example 5.2 Consider the initial value problem (Ezzati and Shakibi 2011)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1 - u^2) \quad (59)$$

with an initial condition of

$$u(x,0) = -0.5 + 0.5\tanh(0.3536x) \quad (60)$$

The exact solution is given by

$$u(x,t) = -0.5 + 0.5\tanh(0.3536x - 0.75t) \quad (61)$$

Example 5.3 We consider the Newell–Whitehead–Segel equation (Nourazar et al. 2011)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 2u - 3u^2 \quad (62)$$

Subject to the initial condition

$$u(x,0) = \lambda \quad (63)$$

The exact solution in a closed form is given by

$$u(x,t) = \frac{-\frac{2}{3}\lambda e^{2t}}{-\frac{2}{3} + \lambda - \lambda e^{2t}} \quad (64)$$

The Newell–Whitehead–Segel equations have wide applicability in mechanical and chemical engineering, ecology, biology and bio-engineering.

Example 5.4 Consider the initial value problem (Hariharan and Kannan 2009a)

$$u_t = u_{xx} + u - u^2 \quad (65)$$

Subject to the initial condition

$$u(x,0) = \frac{1}{\left(1 + e^{\frac{x}{\sqrt{6}}}\right)^2} \quad (66)$$

The exact solution in a closed form is given by

$$u(x,t) = \frac{1}{\left(1 + e^{\frac{x}{\sqrt{6}} - \frac{5}{6}t}\right)^2} \quad (67)$$

Example 5.5 Consider the Newell–Whitehead–Segel equation (Nourazar et al. 2011)

$$u_t = u_{xx} + u - u^4 \quad (68)$$

Subject to the initial condition

$$u(x,0) = \frac{1}{\left(1 + e^{\frac{3x}{\sqrt{10}}}\right)^{\frac{2}{3}}} \quad (69)$$

The exact solution in a closed form is given by

$$u(x,t) = \left(\frac{1}{2}\tanh\left(-\frac{3}{2\sqrt{10}}\left(x - \frac{7}{\sqrt{10}}t\right)\right) + \frac{1}{2}\right)^{\frac{2}{3}} \quad (70)$$

Our results can be compared with Nourazar et al. (2011) results.

Example 5.6 Consider the Newell–Whitehead–Segel equation (Nourazar et al. 2011)

$$u_t = u_{xx} + 3u - 4u^3 \quad (71)$$

Subject to the initial condition

$$u(x,0) = \sqrt{\frac{3}{4}}\frac{e^{\sqrt{6}x}}{e^{\sqrt{6}x} + e^{\frac{\sqrt{6}}{2}x}} \quad (72)$$

The exact solution in a closed form is given by

$$u(x,t) = \frac{\sqrt{3}}{4}\left(\frac{e^{\sqrt{6}x}}{e^{\sqrt{6}x} + e^{\left(\frac{\sqrt{6}}{2}x - \frac{9}{2}t\right)}}\right) \quad (73)$$

Example 5.7: Consider the NW equation (Hammouch and Mekkaoui 2013)

$$u_t = u_{xx} + u - u^3 \quad (74)$$

Subject to the initial condition

$$u(x,0) = \frac{\sinh\left(\frac{x}{\sqrt{2}}\right)}{1 + \cosh\left(\frac{x}{\sqrt{2}}\right)} \quad (75)$$

The exact solution in a closed form is given by

$$u(x,t) = \frac{e^{\frac{x}{\sqrt{2}}} - e^{-\frac{x}{\sqrt{2}}}}{e^{\frac{x}{\sqrt{2}}} + e^{-\frac{x}{\sqrt{2}}} + 2e^{-\frac{3}{2}t}} \quad (76)$$

Example 5.8 Consider the NW equation (Kheiri et al. 2011)

$$u_t = u_{xx} + 4u + 4u^3 \quad (77)$$

Subject to the initial condition

$$u(x,0) = \frac{\left(\cosh\left(\frac{1}{2}\sqrt{2}x\right) + \sinh\left(\frac{1}{2}\sqrt{2}x\right)\right)^2}{4\cosh^2\left(\frac{1}{2}\sqrt{2}x\right)} \quad (78)$$

The exact solution in a closed form is given by

$$u(x,t) = \frac{\left(cosh\left(\frac{1}{\sqrt{2}}\left(x - 3\sqrt{2}t\right)\right) + \sinh\left(\frac{1}{\sqrt{2}}\left(x - 3\sqrt{2}t\right)\right)\right)^2}{4\cosh^2\left(\frac{1}{\sqrt{2}}\left(x - 3\sqrt{2}t\right)\right)} \quad (79)$$

Tables (1, 2, 3, 4, 5) show the numerical solutions of the AC and NW type equations for various values (*x*, *t*). Our LLWM results are in excellent agreement

Table 1 Absolute error of LLWM (Example 5.2)

x	$t = 0.1$	$t = 0.3$	$t = 0.5$
−25	1.18943E−11	7.27843E−12	6.4747E−12
−15	2.36636E−9	9.36464E−10	1.35653E−10
25	9.84744E−10	8.46464E−9	7.49924E−10
30	3.57575E−11	7.484843E−10	2.44443E−10

Table 2 Comparison between the exact and LLWM for Example 5.4

x	t	U_{exact}	U_{LLWM}
0.25	0.5	0.81839	0.81855
	1.0	0.98292	0.98305
	2.0	0.99988	0.99999
	5.0	1.00000	1.00000
0.50	0.5	0.77590	0.77602
	1.0	0.97815	0.97824
	2.0	0.99985	0.99996
	5.0	1.00000	1.0000
0.75	0.5	0.72582	0.72595
	1.0	0.92207	0.92221
	2.0	0.99981	0.99993
	5.0	1.00000	1.00000

Table 3 Comparison between the exact and LLWM for Example 5.6

x	t	Exact	LLWM
0.1	0.2	0.5054	0.5062
0.2	0.4	0.7364	0.7371
0.3	0.6	0.8780	0.8786
0.4	0.8	0.9475	0.9480
0.5	1.0	0.9781	0.9784

Table 4 Numerical values of Example 5.7 ($k = 1$ and $M = 3$)

x	t	*Exact solution* u_{HAM}	*Numerical* u_{LLWM}
0.001	0.001	0.49983572	0.49983571
0.002	0.002	0.49967144	0.49967145
0.003	0.003	0.49950716	0.49950715
0.004	0.004	0.49934288	0.49934288
0.005	0.005	0.49917859	0.49917858
0.006	0.006	0.49901431	0.49901430
0.007	0.007	0.49988501	0.49988502
0.008	0.008	0.49868574	0.49868571
0.009	0.009	0.49852145	0.49852146
0.01	0.01	0.49835716	0.49835712

Table 5 Numerical values of Example 5.8 ($k = 1$ and $M = 3$)

x	$t = 5$
−30	9.8577×10^{-31}
−25	1.3786×10^{-24}
−20	1.9110×10^{-18}
−15	2.6451×10^{-11}
−10	4.4842×10^{-10}

Fig. 1 The surface area shows that $u(x, t)$ using LLWM for Example 5.1 at $x = 0.75$, $k = 3$ and $M = 2$

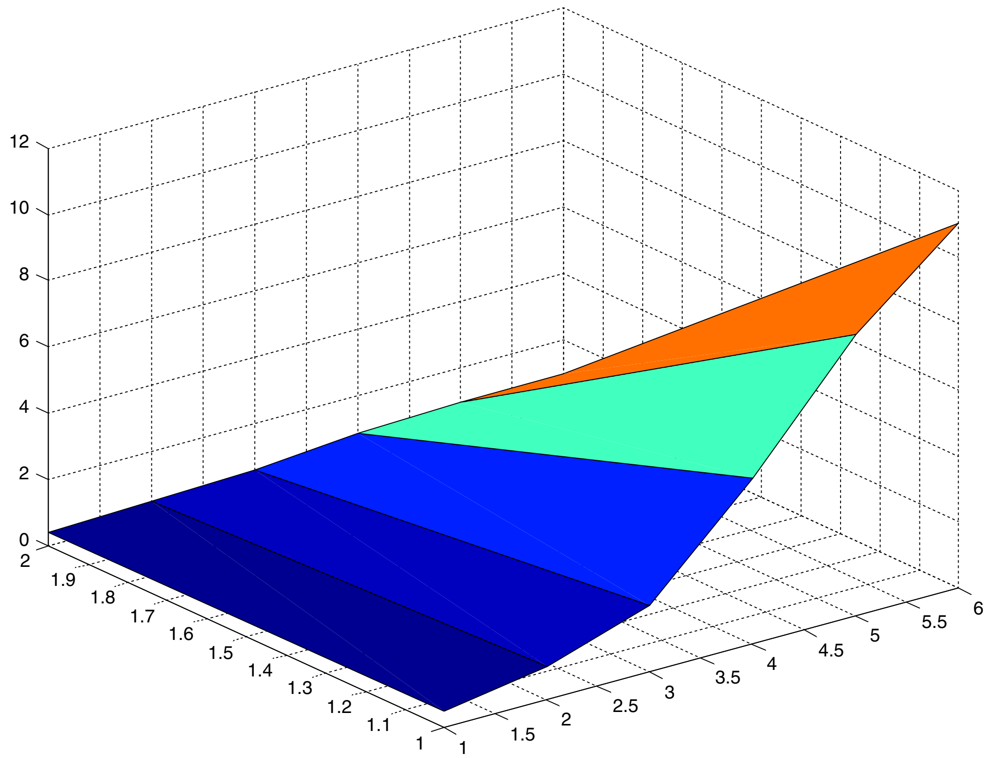
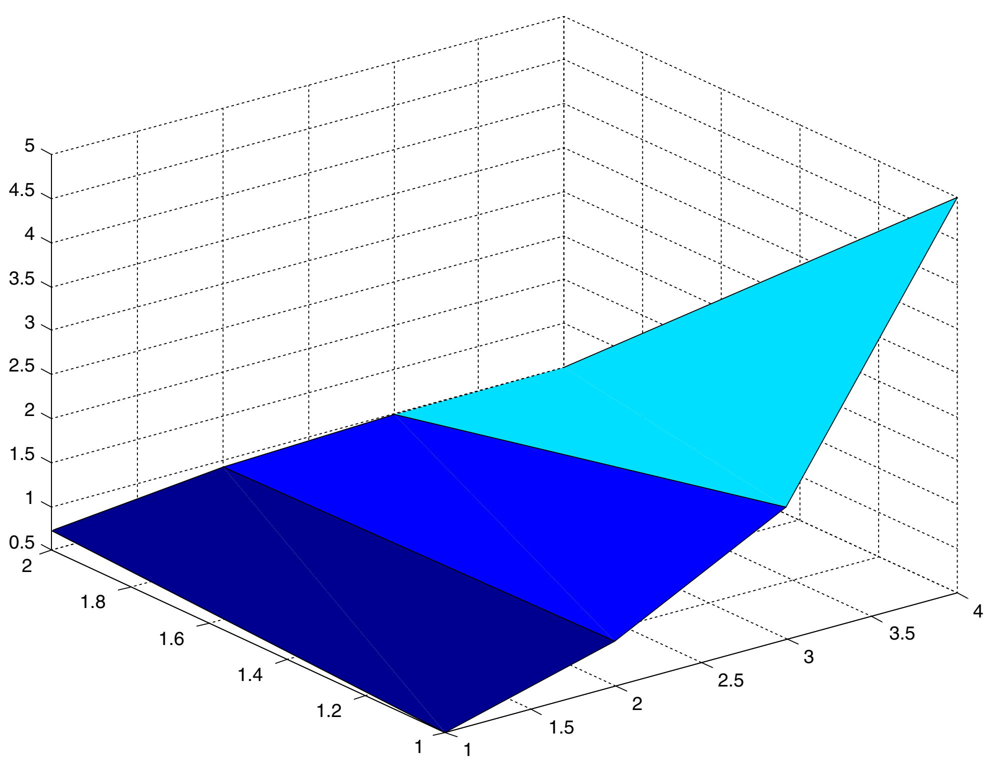


Fig. 2 The surface area shows that $u(x, t)$ using LLWM for Example 5.4 at $x = 0.75$, $k = 2$ and $M = 2$



with the exact solution and those obtained by the Adomian decomposition method (ADM), Homotopy perturbation method (HPM), Homotopy analysis method (HAM) and the differential transform method (DTM). Figs. (1, 2, 3) show the numerical solutions of the AC and NW type equations for various values of (x, t).

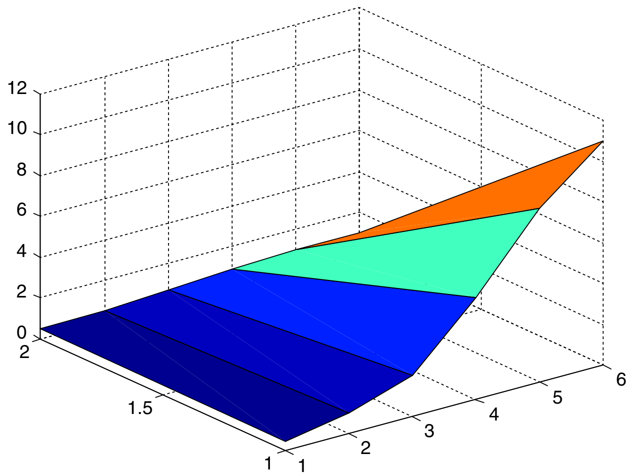


Fig. 3 The surface area shows that $u(x, t)$ using LLWM for Example 5.7 at $x = 0.25$, $k = 2$ and $M = 3$

Good agreement with the exact solution is achieved. For larger values of

All the numerical experiments presented in this section were computed in double precision with some MATLAB codes on a personal computer System Vostro 1400 Processor $\times$ 86 Family 6 Model 15 Stepping 13 Genuine Intel $\sim$596 MHz.

Conclusion

In this work, a new Legendre wavelet-based approximation method has been successfully employed to obtain the numerical solutions of NW and AC equations arising in various fields. The proposed scheme is the capability to overcome the difficulty arising in calculating the integral values while dealing with nonlinear problems. This method shows higher efficiency than the traditional Legendre wavelet method for solving nonlinear PDEs. Numerical example illustrates the powerful of the proposed scheme LLWM. Also this paper illustrates the validity and excellent potential of the LLWM for nonlinear and fractional PDEs. The numerical solutions obtained using the proposed method show that the solutions are in very good coincidence with the exact solution. In addition the calculations involved in LLWM are simple, straight forward and small computation cost. In "Convergence Analysis and Error Estimation" section, we have developed the convergence of the proposed algorithm.

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